2: Lateral Dynamics

PRINTED WITH QUESTION DISCUSSION MANUSCRIPT

2.1: Background

Recommended to read:

- *Gillespie, chapter 6*
- *Automotive Handbook 4th ed., pp 342-353*

The lateral part is planned for in three lectures, each 2x45 minutes:

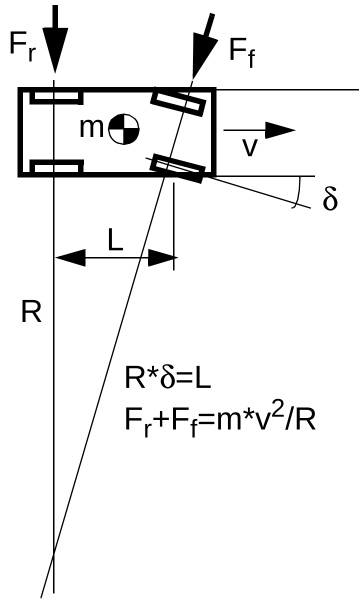
Low speed turning and steady state cornering
Transient cornering
Longitudinal & lateral load distribution during cornering

Theory of Ground Vehicles

X: longitudinal

. . .

Y: lateral



Z: vertical

. . .

Numbers 1.-3. are lecture numbers

steady state handling

1. low speed turning
1. high speed cornering
1 bicycle model, . no states
1. over- & under-steering

transient handling

2. transient cornering
2. bicycle model with two states
2. lane change
2. stability

component & subsystem characteristics

1. Ackermann geometry
1. lateral tire slip

interaction with x direction

3. influences from load distribution

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other courses:

- lateral slope
- steering geometry

- dynamic over- & understeer
- bicycle model with three states
- models with >3 states
- side wind
- closed-loop with human in the loop
- driver models

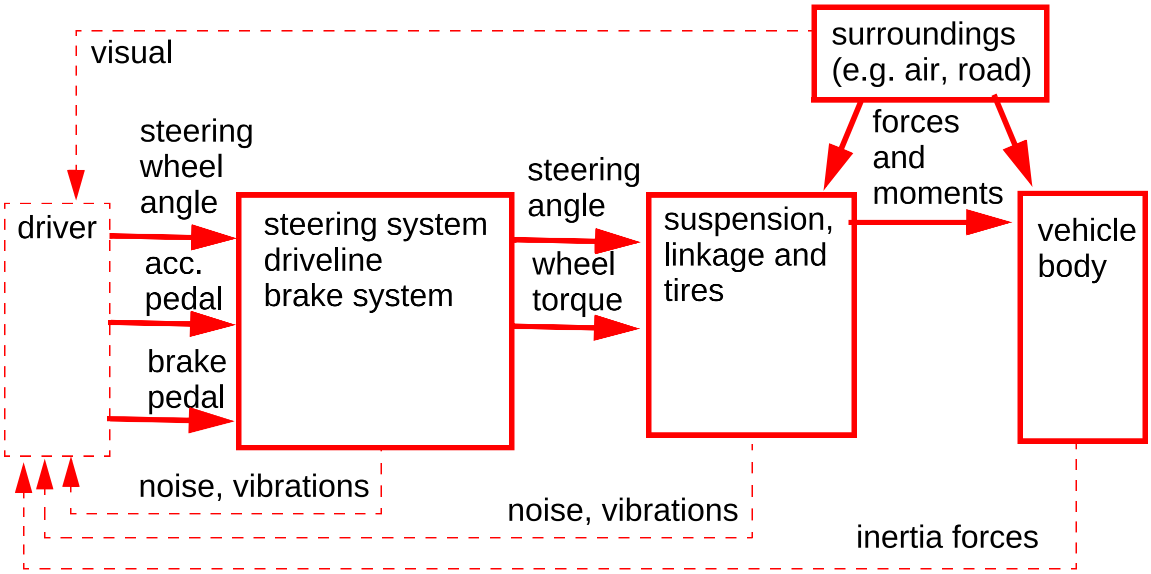
- advanced tire models

- acc./brake in a curve
- “mu-split”
- combined longitudinal & lateral tire slip

Guest lecture: steering of tracked vehicles

2.2: General questions

--- *Question 2.1:* Sketch your view of the open- and closed-loop system, i.e. without and with the driver. As control system block diagram or similar.



open-loop system = only the thick, solid lines
closed-loop system = whole diagram

(NOTE: this kind of diagram is never complete and can always be debated)

Open-loop vs closed loop studies of lateral dynamics. Closed-loop studies involves the driver responsive to feedback in the system. See text in Gillespie, p195, Bosch p 346.

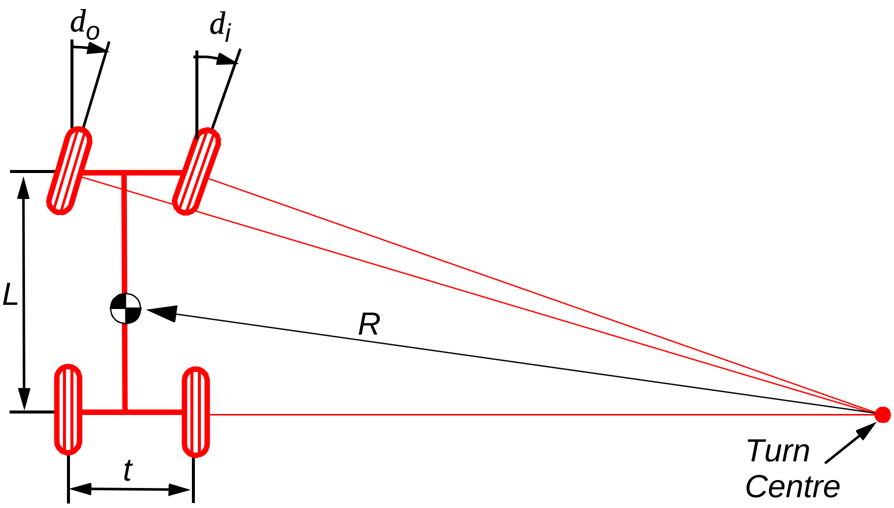
This course will only treat open-loop. How to coop with closed-loop? For example: driver models, simulators or experiments.

2.3: Questions on low speed turning

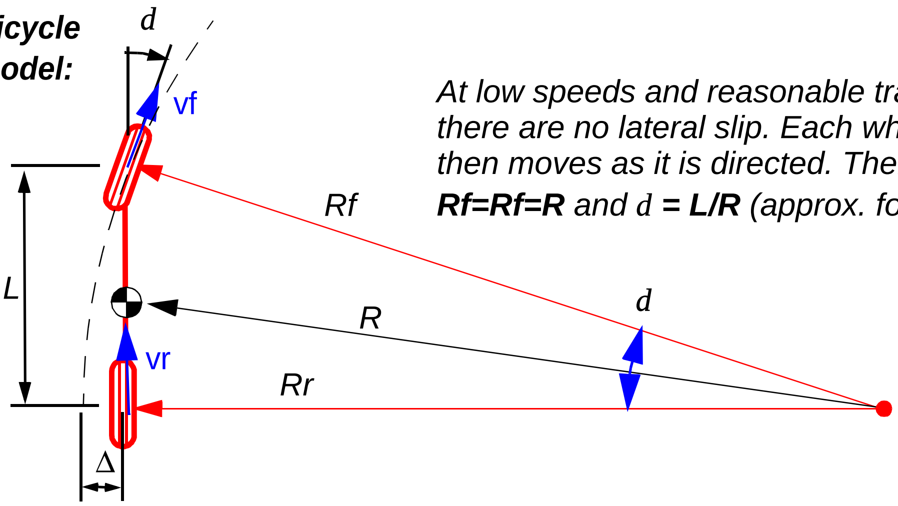
--- *Question 2.2:* Draw a top view of a 4 wheeled vehicle in a turning manoeuvre. How should the wheel steering angles be related to each other for perfect rolling at low speeds?

... Gillespie, fig 6.1. Ackermann steering geometry

Gillespie, fig 6.1: Geometry of a turning vehicle



Bicycle model:

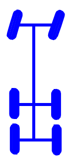


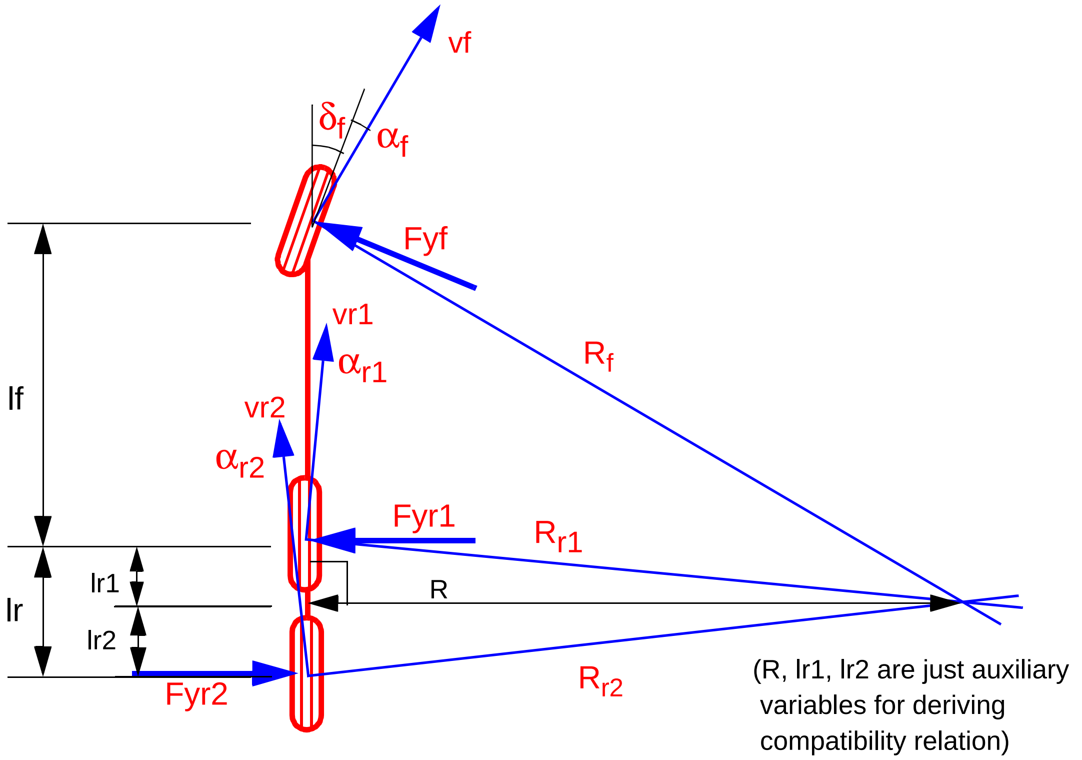
At low speeds and reasonable traction, there are no lateral slip. Each wheel then moves as it is directed. Then:
***Rf=Rf=R** and d **= L/R** (approx. for small angles)*

Off-tracking distance, Δ=R*[1-cos(L/R)]

Deviations from Ackermann geometry affect tire wear and steering system forces significantly but less influence on directional response.

--- *Question 2.3:* Consider a rigid truck with 1 steered front axle and 2 none-steered rear axles. How to predict turning centre?





Assuming: small speed, small steering angles, low traction... But still we cannot assume that each wheel is moving as it is directed. A lateral slip is forced, in a general case at all axles. Turning centre is then not only dependent of geometry, but also forces. The difference compared to the two axle vehicle that we now have 3 unknown forces but only 2 relevant equilibrium equations -- the system is not statically determined.

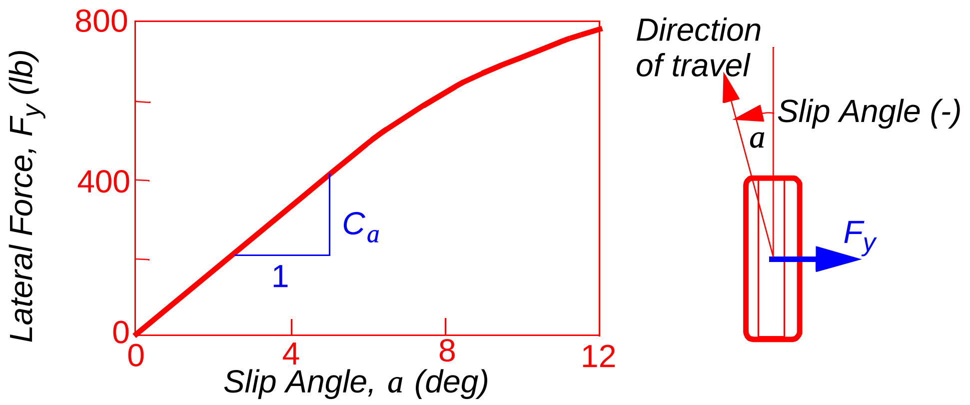
Approx. for small angles:

Equilibrium: Fyf+Fyr1=Fyr2 and Fyf*lf+Fyr2*lr=0

Compatibility: $(\alpha_{r1}+\alpha_{r2})*lf/lr+\alpha_{r1}=\delta_f+\alpha_f$

Constitutive relations: $F_{yf}=C_{\alpha f}*\alpha_f$ $F_{yr1}=C_{\alpha r1}*\alpha_{r1}$ $F_{yr2}=C_{\alpha r2}*\alpha_{r2}$

Gillespie fig 6.2: Tire cornering force properties



Learn C_a = cornering stiffness [N/rad]

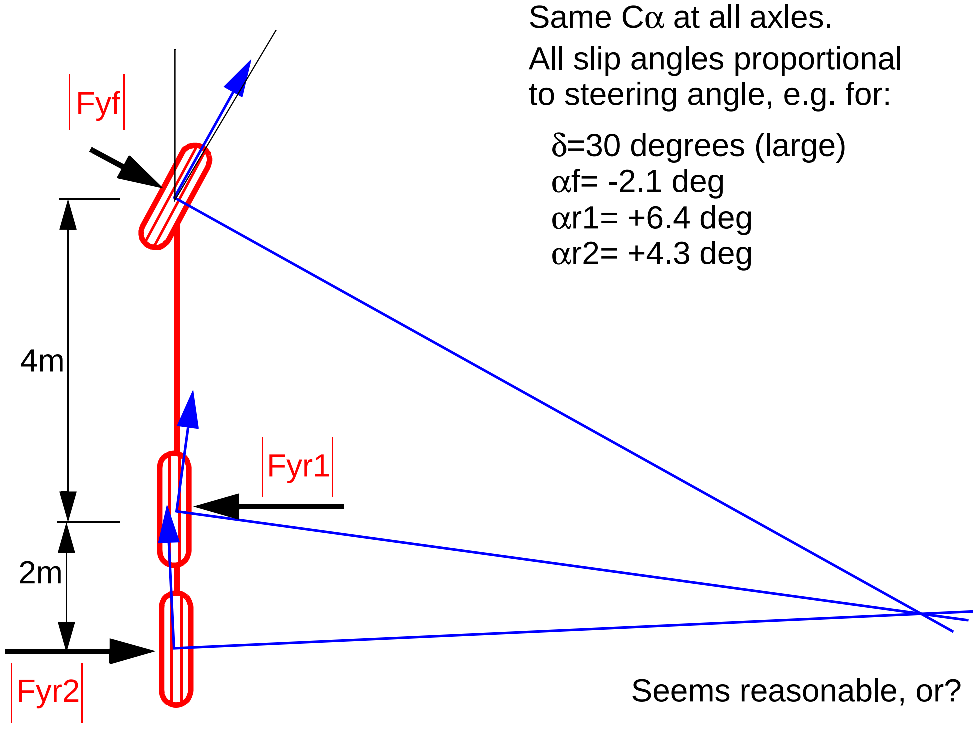
Compare with longitudinal slip:

- s=(R*w-v)/(R*w)=vdiff/vref
- alpha=atan(vy/vx), which is approx. equal to vy/vx=vdiff/vref

More about this in Question 2.16.

Together,3 eqs and 3 unknown (the three slip angles):
$C_{\alpha f}*\alpha_f + C_{\alpha r1}*\alpha_{r1} = C_{\alpha r2}*\alpha_{r2}$ and
$C_{\alpha f}*\alpha_f*lf+C_{\alpha r2}*\alpha_{r2}*lr=0$ and
$lf/(\delta_f + \alpha_f) = lr/(\alpha_{r1} + \alpha_{r2})$

Test, e.g. prescribe steering angle. Calculate slip angles. Have we assumed the correct sense of slip angles? No, not on front axle (αf<0), but on both rear axles (αr1>0 and αr2>0).



Same Cα at all axles.
All slip angles proportional to steering angle, e.g. for:

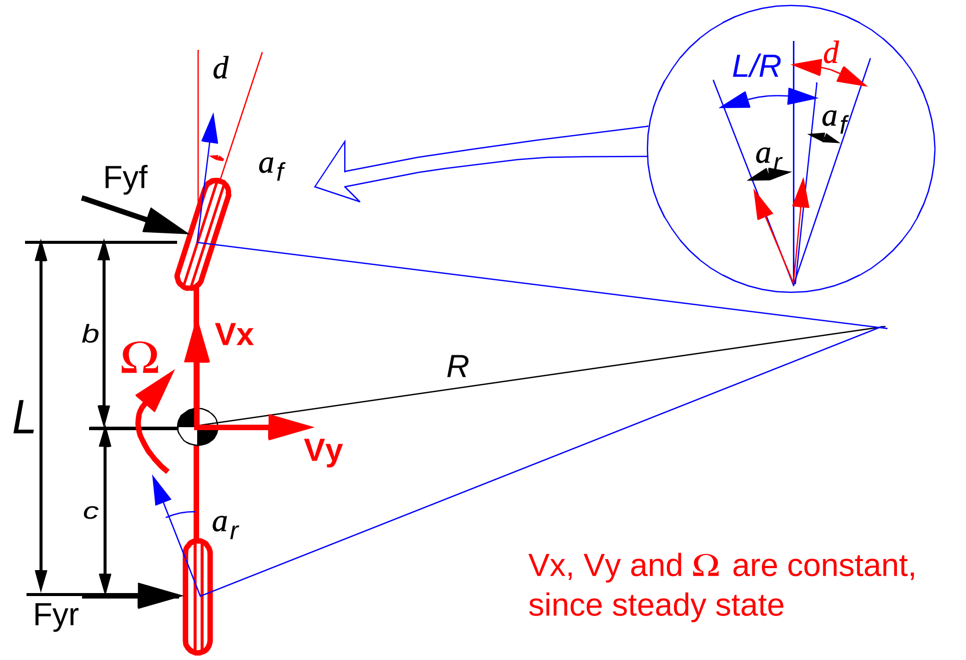
δ=30 degrees (large)
αf= -2.1 deg
αr1= +6.4 deg
αr2= +4.3 deg

Seems reasonable, or?

2.4: Questions on Steady state cornering at high speed

--- *Question 2.4:* In a steady state curve at high speed, centripetal forces is needed to keep the vehicle on the curved track. Where do we find them? How large must these be? How are they developed in practice?

Gillespie, fig 6.4: Cornering of a bicycle model



The centrifugal force = F_c=m*R*Ω^2= m*Vx^2/R. It has to be balance by the wheel/road lateral contact forces:
Equilibrium: Fyf+Fyr = F_c = m*Vx^2/R and Fyf*b-Fyr*c=0

*(Why not Fyf*b-Fyr*c=I*dw/dt ??? Answer: Remember steady state assumed!)*

Constitutive equations: $F_{yf}=C_{\alpha f}*\alpha_f$ and $F_{yr}=C_{\alpha r}*\alpha_r$

Compatibility: tan(d-a_f)=(b*Ω+Vy)/Vx and tan(a_r)=(c*Ω-Vy)/Vx
eliminating Vy for small angles and using Vx=R*Ω: $d - a_f + a_r$=L/R

Together, eliminate slip angles:
Fyf=(lr/L)*m*Vx^2/R and Fyr=(lf/L)*m*Vx^2/R
$d - F_{yf}/C_{\alpha f} + F_{yr}/C_{\alpha r}$=L/R

Eliminate lateral forces:
d = L/R +[(lr/L)/$C_{\alpha f}$ - (lf/L)/$C_{\alpha r}$] * m*Vx^2/R
which also can be expressed as: d = L/R + [Wf/$C_{\alpha f}$ - Wr/$C_{\alpha r}$] *Vx^2/(g*R)
(Wf and Wr are vertical weight load at each axle, respectively.)

Wf/$C_{\alpha f}$ - Wr/$C_{\alpha r}$ is called **understeer gradient** or **coefficient**, denoted **K** or **K_{us}** and simplifies to: d = L/R + K *Vx^2/(g*R)

A more general definition of understeer gradient:

$$K_{us} = \frac{\partial\delta}{\partial a_y}/g \quad \text{[rad/g]}$$

We will learn later, that this relation between d , R and Vx is only the first order theory.

--- *Question 2.5:* Study a 2 axle vehicle in a **low** speed turn. (We return to low speed range temporarily.) How to find steering angle needed to negotiate a turn at a given constant radius. And how does the following quantities vary with steering angle and longitudinal speed:

-- yaw velocity or yaw rate, i.e. time derivative of heading angle

-- lateral acceleration

We will then discuss how it depends on speed at higher speeds.

For a low speed turn:

Needed steering angle: d = L / R (not dependent of speed)

*Yaw rate: W = Vx / R = Vx * d / L (prop. to speed and steering angle)*

*Lateral acceleration: ay = Vx^2 / R = Vx^2 * d / L (prop. to speed and steering angle)*

Since steering angle is the control input, it is natural to define "gains", i.e. division by *d*:

Yaw rate gain = *W/d =Vx/L*

Lateral acceleration gain: *ay/d = Vx^2/L*

For a high speed turn:

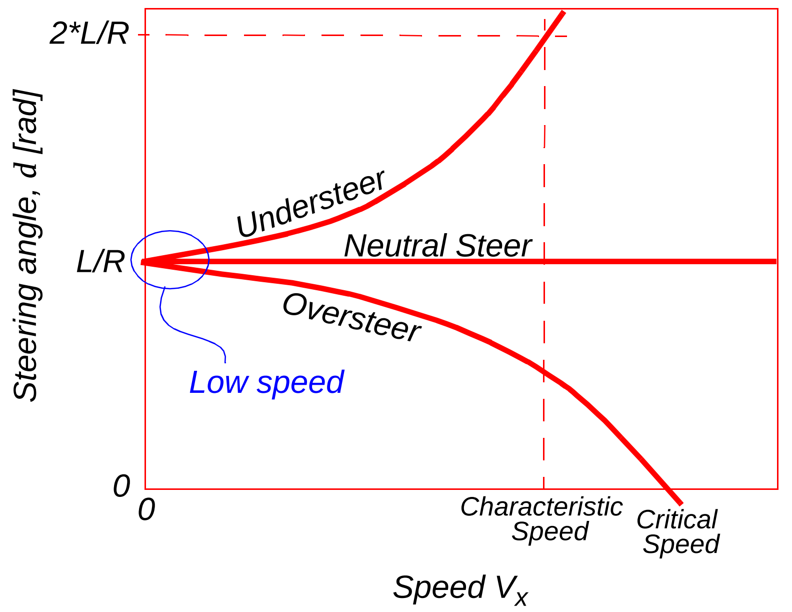
d = L/R + K *Vx^2/(g*R)

Yaw rate gain = *W/d =(Vx/R) / d=Vx/(*L + K *Vx^2/g*)*

Lateral acceleration gain: *ay/d = (Vx^2/R) / d = Vx^2/(*L + K *Vx^2/g*)*

These can be plotted vs Vx:

Gillespie, fig 6.5 (slightly changed): Change of steering angle with speed

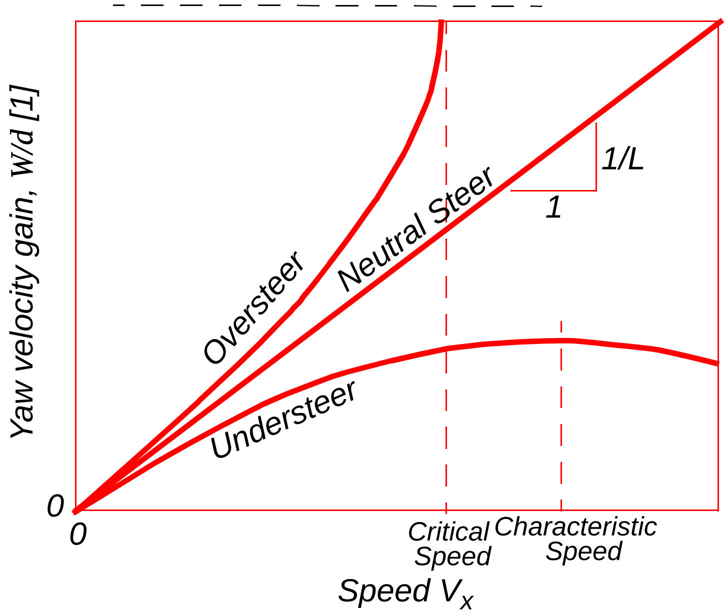


Other plot "curvature gain" or "curvature response" instead, i.e. (1/R)/d

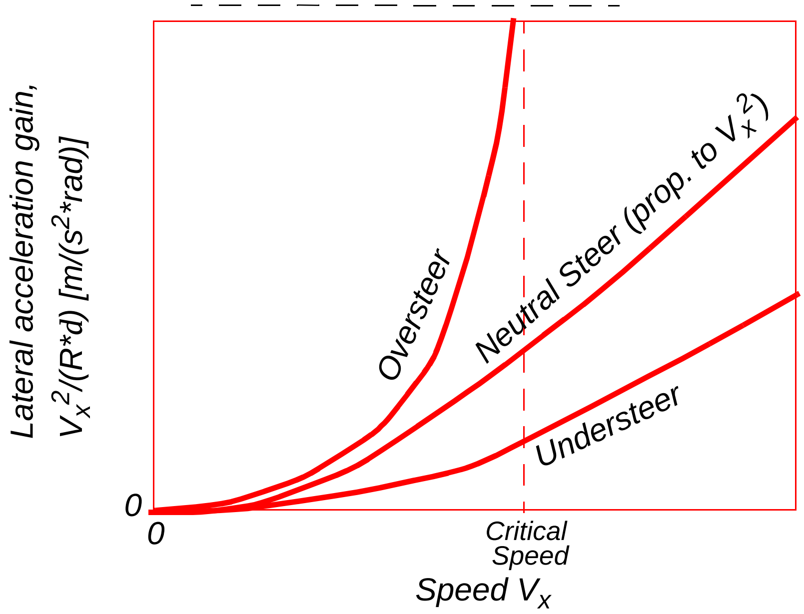
What happens at Critical speed? Vehicle turns in an instable way, even with steering angle=0.

What happens at Characteristic speed? Nothing special, except that twice the steering angle is needed, compared to low speed or neutral.

Gillespie, fig 6.6: Yaw velocity gain as function of speed



Lateral acceleration gain as function of speed (not plotted by Gillespie)

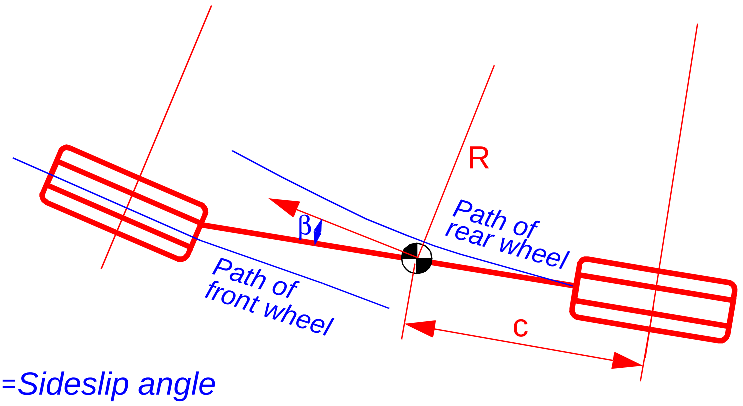


What is the drivers aim when putting up a certain steer angle? At steady state, I would guess, mostly a curvature, maybe a yaw velocity. At transient situations, I would guess, more often a yaw velocity or maybe a lateral acceleration. NOTE: These are my guesses, without being an expert in human cognitive ergonomics.

There are a lot of different definitions of over/understeer. We have used one of the simplest one above.

--- *Question 2.6:* How is the velocity of the centre of gravity directed for low and high speeds?

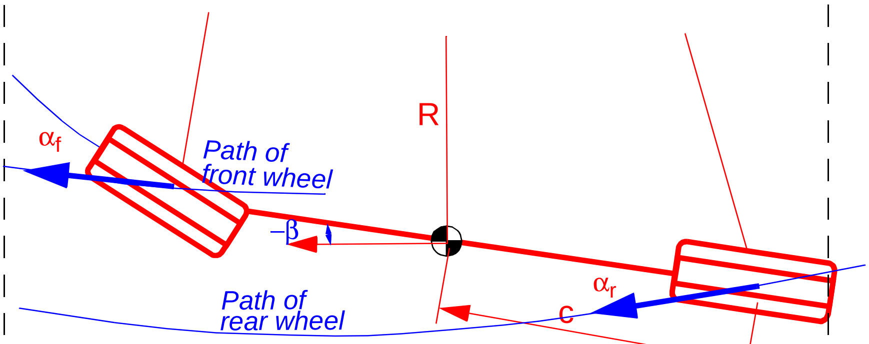
Gillespie, fig 6.7: Sideslip angle in a low-speed turn



b=Sideslip angle

See the differences and similarities between side slip angle for a vehicle and for a single wheel. Bosch calls side slip angle “floating angle”.

Gillespie, fig 6.8: Sideslip angle in a high-speed turn



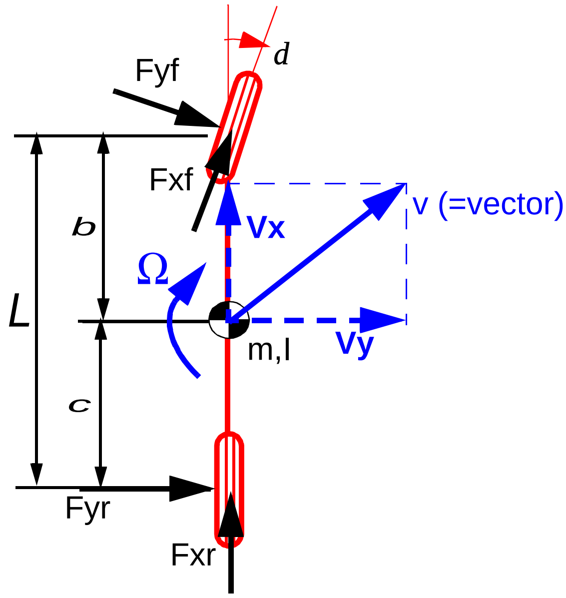
b=Sideslip angle

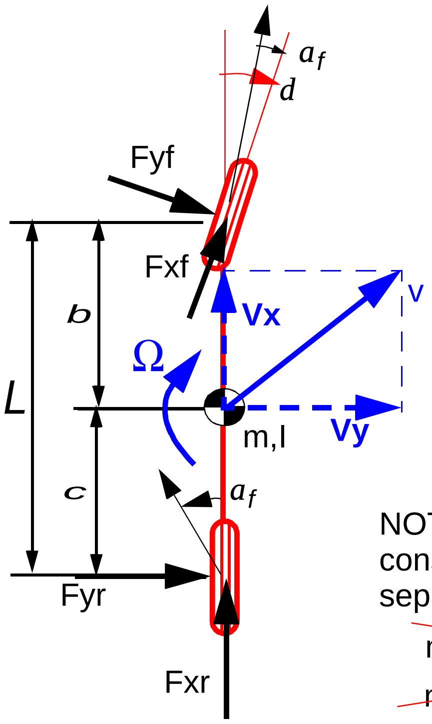
Some (e.g., motor sport journalists) use the word under/oversteer for positive/negative vehicle side slip angle.

2.5: Questions on Transient cornering

NOTE: Transient cornering is not included in Gillespie. This part in the course is defined by the answer in this part of lecture notes. For more details than given on lectures, please see e.g. Wong.

--- *Question 2.7:* To find the equations for a vehicle in transient cornering, we have to start from 3 scalar equations of motion or dynamic equilibrium. Sketch these equations.





v is a vector. Let F also be vectors.

$m \cdot dv/dt = \Sigma F$ (**2D vector equation**)

$I \cdot d\Omega/dt = \Sigma M_z$ (1D scalar equation)

We would like to express all equations as **scalar equations**. We would also like to express it without introducing the heading angle, since we then would need an extra integration when solving (to keep track of heading angle). In conclusion, we would like to use **"vehicle fix coordinates"**.

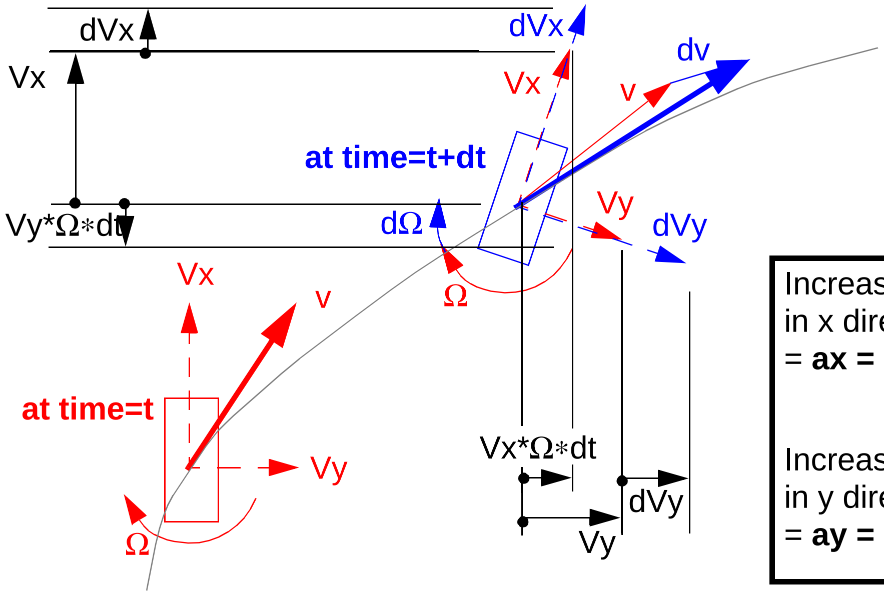
NOTE: It will **NOT** be correct if we only consider each component of v (Vx and Vy) separately, like this:

~~$m \cdot dV_x/dt = Fxr + Fxf \cdot \cos(\delta) - Fyf \cdot \sin(\delta)$~~

~~$m \cdot dV_y/dt = Fyr + Fxf \cdot \sin(\delta) + Fyf \cdot \cos(\delta)$~~

But the torque equation is that straight forward:

$I \cdot d\Omega/dt = -Fyr \cdot c + Fxf \cdot \sin(\delta) \cdot b + Fyf \cdot \cos(\delta) \cdot b$



Increase rate of speed in x direction=
= **ax = dVx/dt - Vy***W

Increase rate of speed in y direction=
= **ay = dVy/dt + Vx***W

Now, it will be correct if:
m***a_x** = m*(**dVx/dt - Vy***W) = Fxr + Fxf*cos(δ) - Fyf*sin(δ)
m***a_y** = m*(**dVy/dt + Vx***W)=Fyr + Fxf*sin(δ) + Fyf*cos(δ)
I*dΩ/dt = - Fyr*c + Fxf*sin(δ)*b + Fyf*cos(δ)*b

Try to understand the difference between (ax,ay) and (dVx/dt,dVy/dt).
*[(ax,ay) are **accelerations**, while (dVx/dt,dVy/dt) are **"changes in velocities"**, as experienced by the driver, which is stuck to the vehicle fix coordinate system]*

Constitutive equations: $F_{yf}=C_{\alpha f}*\alpha_f$ and $F_{yr}=C_{\alpha r}*\alpha_r$

Compatibility: $\tan(d - a_f)=(b*\Omega+Vy)/Vx$ and $\tan(a_r)=(c*\Omega-Vy)/Vx$

Eliminate lateral forces yields:
$m*(dVx/dt - Vy*\Omega) = Fxr + Fxf*\cos(\delta) - C_{\alpha f}*\alpha_f*\sin(\delta)$
$m*(dVy/dt + Vx*\Omega) = C_{\alpha r}*\alpha_r + Fxf*\sin(\delta) + C_{\alpha f}*\alpha_f*\cos(\delta)$
$I*d\Omega/dt = -C_{\alpha r}*\alpha_r*c + Fxf*\sin(\delta)*b + C_{\alpha f}*\alpha_f*\cos(\delta)*b$

Eliminate slip angles yields (a 3 state non linear dynamic model):
$m*(dVx/dt - Vy*\Omega) = Fxr + Fxf*\cos(\delta) - C_{\alpha f}*[d - \text{atan}((b*\Omega+Vy)/Vx)]*\sin(\delta)$
$m*(dVy/dt + Vx*\Omega) =$
$= C_{\alpha r}*\text{atan}((c*\Omega-Vy)/Vx) + Fxf*\sin(\delta) + C_{\alpha f}*[d - \text{atan}((b*\Omega+Vy)/Vx)]*\cos(\delta)$
$I*d\Omega/dt =$
$= -C_{\alpha r}*\text{atan}((c*\Omega-Vy)/Vx)*c + Fxf*\sin(\delta)*b + C_{\alpha f}*[d - \text{atan}((b*\Omega+Vy)/Vx)]*\cos(\delta)*b$

For small angles and dVx/dt=constant, we get the 2 state linear dynamic model:
$m*dVy/dt + [(C_{\alpha f}+C_{\alpha r})/Vx]*Vy + [m*Vx+(C_{\alpha f}*b-C_{\alpha r}*c)/Vx]*\Omega=C_{\alpha f}*d$
$I*d\Omega/dt + [(C_{\alpha f}*b-C_{\alpha r}*c)/Vx]*Vy + [(C_{\alpha f}*b^2+C_{\alpha r}*c^2)/Vx]*\Omega \quad =C_{\alpha f}*b*d$

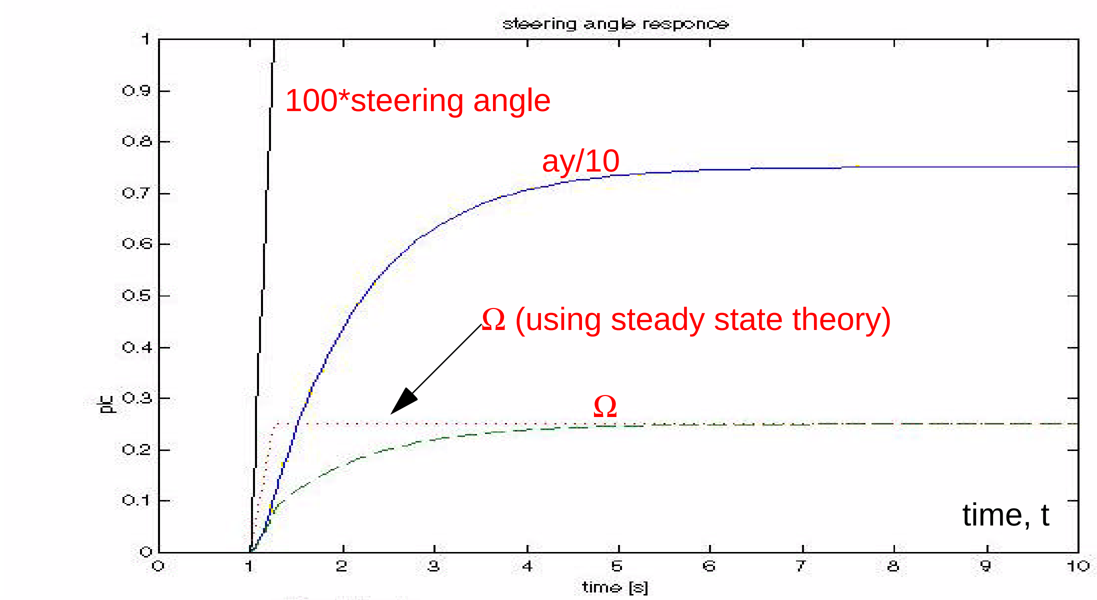
This can be expressed as:

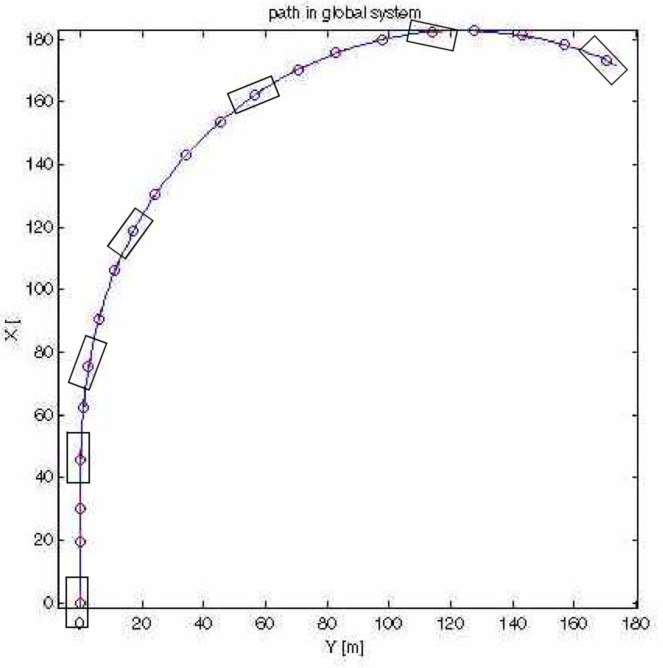
$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} dV_y/dt \\ d\Omega/dt \end{bmatrix} + \begin{bmatrix} \text{4x4 matrix, dependent of} \\ \text{Vx, cornering stiffness and geometry} \end{bmatrix} \cdot \begin{bmatrix} V_y \\ \Omega \end{bmatrix} = \begin{bmatrix} C_{\alpha f} \\ C_{\alpha f} \cdot b \end{bmatrix} \cdot \delta$$

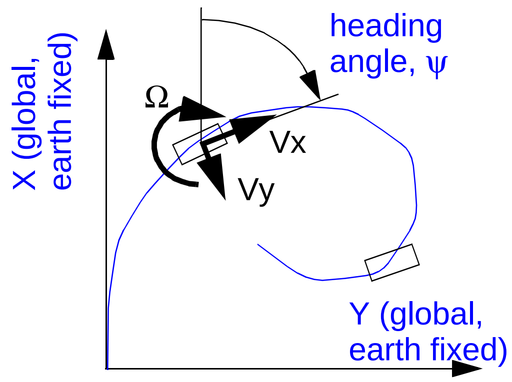
What can we use this for?
- transient response (analytic solutions)
- eigenvalue analysis (stability conditions)

If we are using numerical simulation, there is no reason to assume small angles.

Response on ramp in steering angle:







How to find global coordinates?

$dX/dt = Vx \cdot \cos\psi - Vy \cdot \sin\psi$

$dY/dt = Vy \cdot \cos\psi + Vx \cdot \sin\psi$

$d\psi/dt = \Omega$

Integrate this in parallel during the simulation. Or afterwards, since decouppled in this case.))

More transient tests in Bosch, pp 348-349. Note two types:

- True transients (step or ramp in steering angle, one sinusodal, etc.) (analysed in time domain)
- Oscillating stationary conditions (analysed i frequency domain, transfer functions etc., cf. methods in the vertical art of the course).

Example of variants? Trailer (problem #2), articulated, 6x2/2-truck, all-axle-steering, ...

2.6: Questions on Longitudinal & lateral load distribution during cornering

--- *Question 2.8:* When accelerating, the rear axle will have more vertical load. Explore what happens with the cornering characteristics for each axle. Look at Gillespie, fig 6.3.

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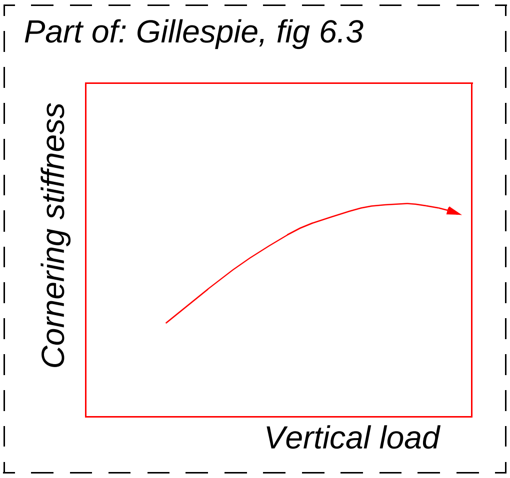

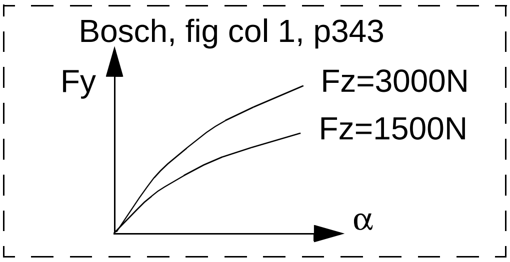

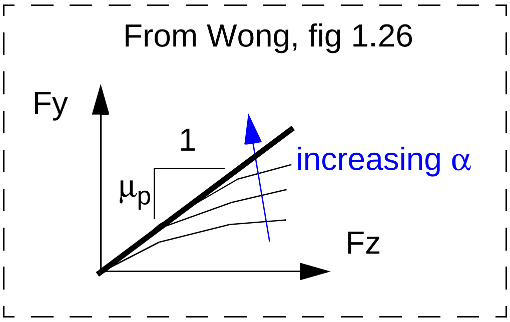

So, the cornering stiffness will ***increase at rear axle*** *and decrease at front axle, due to the longitudinal vertical load distribution at* ***acceleration****.*

This means less tendency for the rear to drift outwards in a curve (and increased tendency for front axle), when accelerating.

(The opposite reasoning by decceleration.)

Learn: Cornering coefficient CC_α [1/rad]: **$CC_a=C_a/Fz$**

So, longitudinal distribution of vertical loads influence handling properties.

NOTE: A larger influence is ften found from the **combined longitudinal and lateral slip** which occurs due to the traction force needed to accelerate.

--- *Question 2.9:* In a curve, the outer wheels will have more vertical load. Explore what happens with the lateral force on an axle, for a given slip angle, if vertical load is distributed differently to left and right wheel. Look at Gillespie, fig 6.11.

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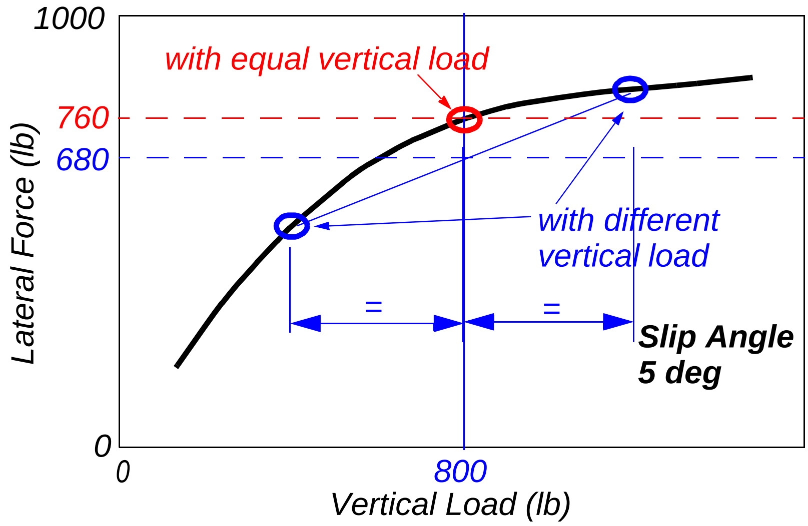
Gillespie, fig 6.11: Lateral force-vertical load characteristics of tires.



*So, the lateral force for the axle will decrease from 2*760 to 2*680.*

So, lateral distribution of vertical loads influence handling properties.

--- *Question 2.10:* How to calculate the vertical load on front and rear axles, respectively, when the vehicle accelerates?

In general: $\Sigma Fz=mg$ and $\Sigma Fz,rear=mg/2+(h/L)*m*ax$, where L=wheel base and h=centre of gravity height. ax=longitudinal acceleration. Still valid for braking because ax is then negative.

--- *Question 2.11:* How to calculate the vertical load on inner and outer side wheels, respectively, when the vehicle goes in a curve?

In general: $\Sigma Fz=mg$ and $\Sigma Fz,outer=mg/2+(h/B)*m*ay$, where B=track width and h=centre of gravity height. ay=lateral acceleration, which is Vx^2/R for steady state cornering.

--- *Question 2.12:* How is vertical load distributed between front/rear, if we know distribution inner/outer?

It depends on roll stiffness at front and rear. Using an extreme example, without any roll stiffness at rear, all lateral distribution is taken by the front axle. So in that case we have: Fz,f,outer=mg/2+(h/B)*m*ay and Fz,r,outer=Fz,r,inner=mg/2.

In a more general case:

Total roll moment = Mx = h*m*ay = Mxf+Mxr
Roll moment on front axle = Mxf = (Fz,f,outer-Fz,f,inner)*B/2 and
Roll moment on rear axle = Mxr = (Fz,r,outer-Fz,r,inner)*B/2

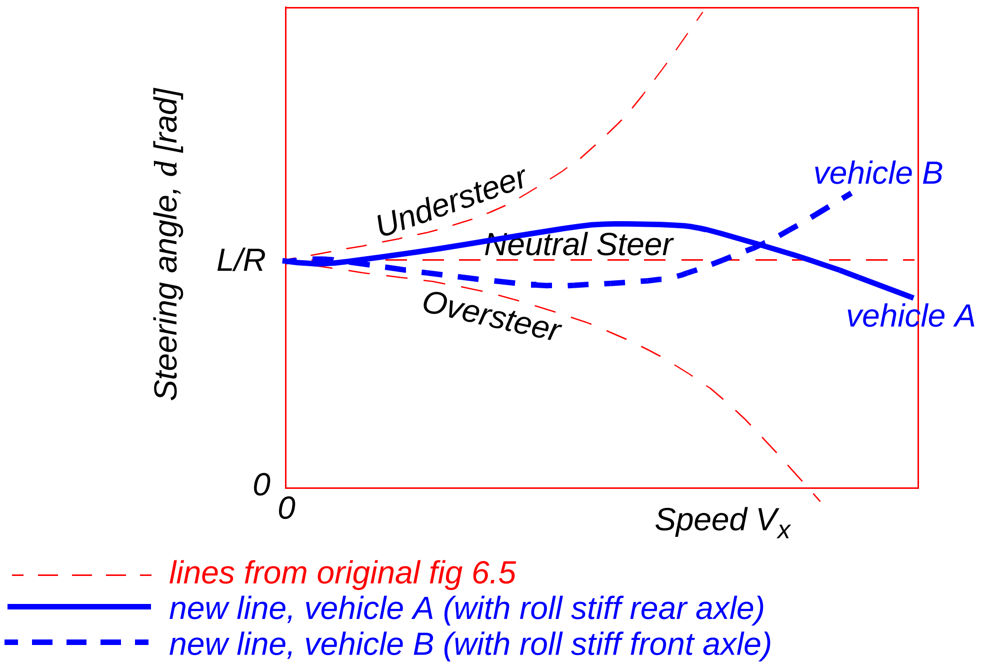
$Mxf=kf*\phi$
$Mxr=kr*\phi$, where kf and kr are roll stiffness and ϕ=roll angle.

Eliminating roll angle tells us that Mxf=kf/(kf+kr)*Mx and Mxr=kr/(kf+kr)*Mx, i.e. the roll moment is distributed proportional to the roll stiffness between front and rear axle. The we can express each Fz in m*g, ay, geometry and kf/kr. This is treated in Gillespie, page 211-213.

--- *Question 2.13:* How would the diagrams in Gillespie, fig 6.5-6.6 change if we include lateral load distribution in the theory?

It results in a nw function δ=func(Vx), (eq 6-48 combined with 6-33 and 6-34). It could be used to plot new diagrams like Gillespie, fig 6.5-6.6:

Redrawn version of *Gillespie, fig 6.5: Change of steering angle with speed*



Equations to plot these curves are found in Gillespie, pp 214-217. Gillespie uses the non linear constitutive equation: $Fy=C_{\alpha}*\alpha$ where $C_{\alpha}=a*Fz-b*Fz^2$.

--- *Question 2.14:* What more effects can change the steady state cornering characteristics for a vehicle at high speeds?

See Gillespie, pp 209-226: E.g. Roll steer and tractive (or braking!) forces. Braking in a curve is a crucial situation. Here one analyses both road grip, but also combined dive and roll (so called warp motion).

--- *Question 2.15:* Try to think of some empirical ways to measure the curves in diagrams in Gillespie, fig 6.5-6.6.

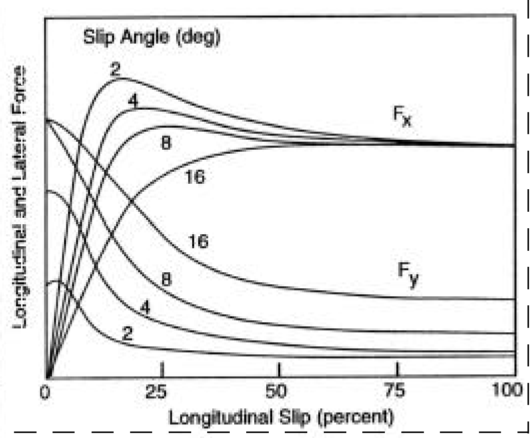
See Gillespie, pp 27-230:

- Constant radius
- Constant speed
- Constant steer angle (not mentioned in Gillespie)

2.7: Questions for component characteristics

--- *Question 2.16:* Plot a curve for constant side slip angle, e.g. 4 degrees, in the plane of longitudinal force and lateral force. Do the same for a constant slip, e.g. 15%. Use Gillespie, fig 10.22 as input.

Gillespie, fig 10.22: Brake and lateral forces as function of longitudinal slip



See Gillespie fig 10.23

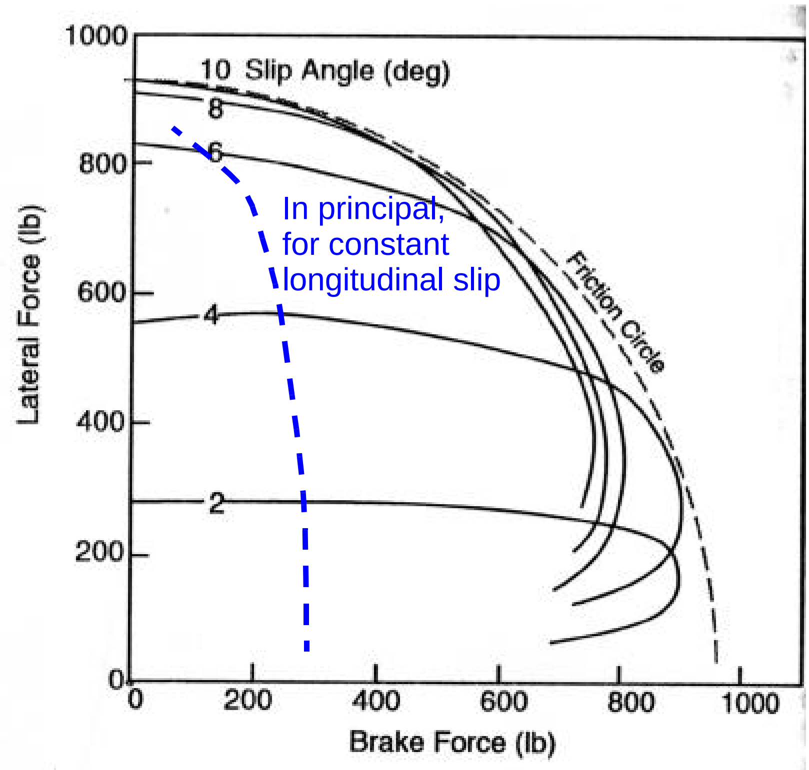


Fig. 10.23 Lateral force versus longitudinal force at constant slip angles.

2.8: Summary

- **low speed turning**: slip only if none-Ackermann geometry
- **steady state cornering at high speeds**: always slip, due to centrifugal acceleration of the mass, $m*v^2/R$
- **transient handling at constant speed**: always slip, due to all inertia forces, both translational mass and rotational moment of inertia
- **transient handling with traction/braking**: not really treated, except that the system of differential equations was derived (before linearization, when Fx and dVx/dt was still included)
- **load distribution, left/right, front/rea**r: We treated influences by steady state cornering at high speeds. Especially effects from roll moment distribution.

Recommended exercise on your own: Gillespie, example problem 1, p 231. (If you try to determine “static margin”, you would have to study Gillespie, pp 208-209 by yourself.)

In the problem you will perform:

- Predict and verify steady state handling characteristics for a car
- Predict and verify transient handling characteristics for a car
- Predict transient handling characteristics for a car with trailer

You will learn and use the following tools:

- Bicycle model
- Solve initial value problem using Matlab’s built-in ode functions
- Extended bicycle model (car with trailer)
- Experimental techniques